

SUB-SEMI-RIEMANNIAN GEOMETRY ON H -TYPE GROUPS

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ABSTRACT. We consider H (eisenberg)-type groups whose law of left translation gives rise to a bracket generating distribution of step 2. In the contrast with sub-Riemannian studies we furnish the horizontal distribution with a nondegenerate indefinite metric of arbitrary index and investigate the problem concerning causal geodesics on underlying manifolds. The exact formulae for geodesics are obtained.

1. INTRODUCTION

Sub-semi-Riemannian geometry is a study of nonholonomic manifolds M whose non-integrable tangent subbundles \mathcal{D} of the tangent bundle TM are equipped with a nondegenerate indefinite metric g (sub-semi-Riemannian metric). If the metric g on \mathcal{D} is a positively definite metric, then the triple (M, \mathcal{D}, g) is known as sub-Riemannian manifold which has become an actively studied mathematical topic. If g is a nondegenerate indefinite metric on \mathcal{D} of index $p \in \mathbb{N}$, then the corresponding object can be called a sub-semi-Riemannian manifold; if $p = 1$, the manifold is called sub-Lorentzian. If \mathcal{D} coincides with the entire tangent bundle TM and g is nondegenerate metric of index greater than 1, then the corresponding triplet reduces to semi-Riemannian manifold. There are numerous applications of nonholonomic manifolds with positively definite metric in problems in physics of high energy, control theory, electromagnetism, mechanical and biological systems with nonholonomic constraints. Therefore, it is interesting to start studying more generalized objects – sub-semi-Riemannian manifolds.

Most of notions in sub-semi-Riemannian geometry originate in sub-Riemannian geometry [17, 3] and Lorentzian geometry [18].

The intention of the article is to enlighten some aspects of sub-semi-Riemannian geometry. We start with a study of geometry, analysis and underlying algebraic structures of H (eisenberg)-type groups with nondegenerate metric. These 2-step nilpotent Lie groups are interesting and natural examples of nonholonomic manifolds in sub-semi-Riemannian sense and at the same time first large class of objects with similar behavior which can be described in sub-semi-Riemannian geometry. The notion of H -type groups with positively definite metric was first introduced by A. Kaplan in his works [9, 10, 11] with the help of theory of Clifford algebras. Such groups with a Riemannian metric provide nice examples of nonholonomic manifolds to study for a sub-Riemannian geometry [4, 5].

Some examples of H -type groups with semi-Riemannian metric on the distribution were already considered in [8, 12, 13, 14] and this is the first attempt to obtain a generalizing result concerning such geometric objects of interest as geodesics. We solve the geodesic equation for sub-semi-Riemannian geodesics, which coincides with the geodesic equation in sub-Riemannian

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case if the considered metric tensor is Riemannian. The essential role here plays the skew-symmetry of emerging objects with respect to the semi-Riemannian product on V , which is not positively definite anymore (see the section 4). The skew-symmetry arises from the use of Clifford algebras and their representations. A good deal of the article is devoted to the description of skew-symmetric transformations w. r. t. semi-Riemannian metric. The canonical view and, therefore, eigenvalues of corresponding matrices that are skew-symmetric w. r. t. semi-Riemannian metric g greatly depends on the index of g .

The article is organized in the following way. In Section 2 we introduce H -type groups endowed with semi-Riemannian metric. We also present necessary basic notions from space-time geometry. In Section 3 we introduce exponential map, with its help we calculate left-invariant vector fields and the connection. Section 4 is devoted to calculations of geodesics and auxiliary algebraic backgrounds. And in Section 5 we provide the example of H -type group with semi-Riemannian metric which illustrates how the expounded theory could be applied for the problem of finding geodesics in a particular case. We conclude the survey by the list of open problems in Section 6.

2. GENERAL STRUCTURE OF HEISENBERG-TYPE GROUPS WITH NONDEGENERATE METRIC

Let U and V be vector spaces of dimensions m and n respectively with positively definite product $\langle \cdot, \cdot \rangle_U$ on U and nondegenerate product $\langle \cdot, \cdot \rangle_V$ of signature (p, q) , $p + q = n$, on V . Denote by η the metric tensor of the product on V , which is a diagonal matrix with first p negative unities and then q positive unities, i. e.

$$(2.1) \quad \eta = \begin{pmatrix} -I_p & 0 \\ 0 & I_q \end{pmatrix},$$

where I_p and I_q are usual identity matrices of dimensions $(p \times p)$ and $(q \times q)$ respectively. The corresponding metric $g: V \times V \rightarrow \mathbb{R}$ is called a nondegenerate metric of index p , $p \in \mathbb{N}$. Then as usually $g(v, w) = \langle v, w \rangle_V = w^t \eta v$, where $v, w \in V$ and w^t denotes the vector transposed to w . Nondegeneracy is understood here in the sense that if $\langle v, w \rangle_V = 0$ for any $w \in V$, then $v = 0$. If the index of the metric is 1, then it is called the Lorentzian metric. The metric with the index 0 is just a Riemannian metric. In the case of arbitrary index such a nondegenerate indefinite metric received the name semi-Riemannian metric.

Let us denote by the symbol $\text{End}(V)$ the set of linear maps from V to V . Consider a linear mapping $j: U \rightarrow \text{End}(V)$, which satisfies the following properties:

$$(2.2) \quad \begin{aligned} i) \quad & j^2(u) = -|u|_U^2 I_V, \\ ii) \quad & \langle \eta j(u)v, w \rangle_V = -\langle v, \eta j(u)w \rangle_V, \end{aligned}$$

where $u \in U$, $v, w \in V$ and I_V is a $(n \times n)$ identity matrix in V , $|u|_U^2 = \langle u, u \rangle_U$. Conditions (2.2) are taken from [11] and adapted for the case of semi-Riemannian metric on V . For all $u \in U$ the linear map $j(u)$ provides a complex structure on V compatible with its euclidean structure. By abuse of notations we use the symbol j to denote both the mapping and the corresponding $(n \times n)$ -matrix.

Remark. The property of j being skew-symmetric remains the same as in the case of H -type groups with positively definite metrics on both spaces U and V ([11]). Indeed, notice that

$$\begin{aligned} \langle \eta j(u)v, w \rangle_V &= w^t \eta j(u)v = w^t j(u)v, \\ \langle v, \eta j(u)w \rangle_V &= w^t j(u)^t \eta v = w^t j(u)^t v \end{aligned}$$

and from the condition *ii*) it follows that

$$(2.3) \quad j(u) = -j(u)^t.$$

Then we can define the bilinear skew-symmetric map $[\cdot, \cdot]: V \times V \rightarrow U$ by

$$(2.4) \quad \langle u, [v, w] \rangle_U = \langle \eta j(u)v, w \rangle_V.$$

We can now define a Lie algebra

$$\mathcal{G} = U \oplus V$$

with a Lie structure on it:

$$[u + v, r + w] = [v, w] \quad (u, r \in U, v, w \in V).$$

Constructed in this way Lie algebra is graded and 2-step nilpotent with the center U . As in [11] we shall refer to these algebras as H -type algebras with nondegenerate metric $\langle \cdot, \cdot \rangle_V$.

Recall that if U is a m -dimensional real space \mathbb{R}^m with a basis consisting of m orthogonal unit vectors e_1, \dots, e_m , then the Clifford algebra $\text{Cl}(U, m)$ is the real associative algebra generated by the set $\{e_1, \dots, e_m\}$, satisfying the relations (see [16])

$$\begin{aligned} e_\alpha^2 &= -1, \quad \alpha = 1, \dots, m, \\ e_\alpha e_\beta &= -e_\beta e_\alpha, \quad \text{for } \alpha \neq \beta. \end{aligned}$$

The Clifford algebra $\text{Cl}(U, m)$ is 2^m -dimensional with the following basis:

1	the scalar
e_1, \dots, e_m	vectors
$e_1 e_2, \dots, e_{m-1} e_m$	bivectors
$e_1 e_2 e_3, \dots, e_{m-2} e_{m-1} e_m$	3-vectors
\vdots	\vdots
$e_1 e_2 \dots e_m$	the m -volume element.

Hence an arbitrary element of the basis may be written as $e_{\mathcal{A}} = e_{\alpha_1} \dots e_{\alpha_i}$; here $\mathcal{A} = (\alpha_1, \dots, \alpha_i)$ and $1 \leq \alpha_1 < \alpha_2 < \dots < \alpha_i \leq m$. All elements of this type share the basic property of the imaginary unit of being the square root of -1 , i. e. $(e_{\mathcal{A}})^2 = -1$. It is known that real Clifford algebras $\text{Cl}(U, m)$ are isomorphic to real matrix algebras with entries in \mathbb{R} , \mathbb{C} , \mathbb{H} or in $\mathbb{R} \oplus \mathbb{R}$, $\mathbb{H} \oplus \mathbb{H}$, where by \mathbb{R} , \mathbb{C} and \mathbb{H} we denote division rings of real numbers, complex numbers and of quaternions respectively.

The mapping $j: U \rightarrow \text{End}(V)$ satisfying (2.2) extends to a representation of Clifford algebra $\text{Cl}(U, m)$ on V ; in other words, the space V becomes a Clifford module over $\text{Cl}(U, m)$. The images of the generators e_1, \dots, e_m of a Clifford algebra $\text{Cl}(U, m)$ under a representation $j: j_\alpha := j(e_\alpha)$, $\alpha = 1, \dots, m$, are called (*standard*) *generators* of the representation. It was noticed that without loss of generality one can represent standard generators as a set of anti-commuting skew-symmetric matrices with squares $-I_V$ (see [2]). Moreover, these matrices can be expressed as homogeneous tensor products of the standard Pauli (2×2) -matrices, though the determination of such matrices is nontrivial.

Recall also that V admits a structure of Clifford module over $\text{Cl}(U, m)$ if and only if the following condition on dimensions n and m of vector spaces V and U , respectively, holds: $0 \leq m < \rho(n)$, where ρ is the Hurwitz-Radon function defined by $n = k2^{4r+s} \mapsto \rho(n) = 8r + 2^s$, k – odd, $0 \leq s \leq 3$ [11]. If V is a normed division algebra, then $U = \text{Im}V$ is a space of imaginary elements of V and $\dim U = m = n - 1$. If $m = 0$ and n is arbitrary, then \mathcal{G} is the Euclidean n -dimensional space; if $m = 1$ and n is even, then \mathcal{G} is the Heisenberg algebra; if $m = 3$ and $n = 4k$, then \mathcal{G} is the Quaternion algebra; if $m = 7$ and $n = 8$, then we call the corresponding algebra an $\mathbb{H} \oplus \mathbb{H}$ -algebra. These four algebras generate a special class among the H -type

algebras, satisfying the j^2 -condition: for all $u_1, u_2 \in U$ with $\langle u_1, u_2 \rangle_U = 0$ and all nonzero $v \in V$ there exists a $u_3 \in U$ so that

$$j(u_1)j(u_2)v = j(u_3)v.$$

Exact formulae for geodesics on some of corresponding groups with semi-Riemannian metric on V were found in [12, 13, 14]. In the present article we concentrate our attention on the general H -type groups with nondegenerate metric on V .

Let us consider the H -type group G with the Lie algebra $\mathcal{G} = V \oplus U$ and nondegenerate product $\langle \cdot, \cdot \rangle_V$ on V . The Lie algebra \mathcal{G} is identified with the tangent space to G at the identity $T_e G$. The scalar product $\langle \cdot, \cdot \rangle$ on the algebra \mathcal{G} is a sum of the inner product $\langle \cdot, \cdot \rangle_U$ on U and nondegenerate product $\langle \cdot, \cdot \rangle_V$ on V . Push-forward allows to define the scalar product on the whole tangent bundle $TG = \cup_\sigma T_\sigma G$, not only in $T_e G$:

$$\langle \cdot, \cdot \rangle(\sigma) = \langle dL_{\sigma^{-1}\cdot}, dL_{\sigma^{-1}\cdot} \rangle(e),$$

where L_σ denotes the left translation on G by the element $\sigma \in G$. Since $V \subset T_e G$, the left translation of V by element σ is $dL_\sigma(V) \subset T_\sigma G$. The mapping $\mathcal{D}: \sigma \mapsto dL_\sigma(V)$ is called a *horizontal distribution* and any vector $v(\sigma) \in dL_\sigma(V)$ is called a *horizontal vector* at point σ .

By analogy with sub-Riemannian geometry the absolutely continuous curve $c(s): [0, 1] \rightarrow \mathcal{D}$ is called the *horizontal curve* if the tangent vector $\dot{c}(s)$ belongs to \mathcal{D} at each $s \in [0, 1]$. The Lie group G was introduced in such a way, that the horizontal distribution \mathcal{D} possesses a *2-step bracket generating property*. It means that for any point $p \in G$ we have that any tangent vector can be presented as a linear combination of the vectors of the following types $v(p)$, $[v, w](p) \in T_p M$, where all vectors v, w are horizontal. A result of Chow [6] says that any two points on G can be connected by a piecewise smooth horizontal curve because of the bracket generating property of \mathcal{D} . Chow theorem guarantees that the connectivity of a manifold by curves tangent to a given distribution \mathcal{D} does not depend neither on metric defined on it nor on tangent bundle.

In this article we study the triple (G, \mathcal{D}, g) , which we call the *sub-semi-Riemannian manifold*, where the nondegenerate bundle-type metric g of arbitrary index p is given on distribution \mathcal{D} . We will call the corresponding group G sub-semi Riemannian H -type group. Due to left translation we can work further with the distribution at the point e , i. e. with $dL_e(V) = V$ instead of \mathcal{D} .

The nondegeneracy of the metric g brings the causal character of the vectors in V . The notations are carried from the Lorentzian geometry [1]. Even though the metric g has the arbitrary index p and the time orientation can not be defined in general (unless $p = 1$), we can anyway introduce the notions of timelikeness, spacelikeness and lightlikeness. A horizontal vector $v \in V$ is called *timelike* if $\langle v, v \rangle_V < 0$, *spacelike* if $\langle v, v \rangle_V > 0$ or $v = 0$, *lightlike* if $\langle v, v \rangle_V = 0$ and $v \neq 0$, *nonspacelike* if $\langle v, v \rangle_V \leq 0$. A horizontal curve is called timelike if its tangent vector is timelike at each point; similarly for spacelike, lightlike and nonspacelike.

3. EXPONENTIAL MAP, LEFT-INVARIANT VECTOR FIELDS AND CONNECTION

We define the exponential map as a function $\exp: \mathcal{G} \rightarrow G$ in a usual way in the Lie groups theory. Then we can define smooth maps $\mathbf{u}: G \rightarrow U$ and $\mathbf{v}: G \rightarrow V$ such that

$$\sigma = \exp(\mathbf{u}(\sigma) + \mathbf{v}(\sigma)) \quad \text{for } \sigma \in G.$$

Since \mathcal{G} is a 2-step nilpotent Lie algebra, then the Baker-Campbell-Hausdorff formula yields

$$(3.1) \quad \begin{aligned} \mathbf{v}(\sigma_1 \sigma_2) &= \mathbf{v}(\sigma_1) + \mathbf{v}(\sigma_2), \\ \mathbf{u}(\sigma_1 \sigma_2) &= \mathbf{u}(\sigma_1) + \mathbf{u}(\sigma_2) + \frac{1}{2}[\mathbf{v}(\sigma_1), \mathbf{v}(\sigma_2)]. \end{aligned}$$

Suppose that $\{\partial_{v_j}\}_{j=1}^n$ and $\{\partial_{u_\alpha}\}_{\alpha=1}^m$ are orthonormal bases for V and U respectively such that $\langle \partial_{v_i}, \partial_{v_j} \rangle_V = \varepsilon_j \delta_{ij}$ and $\langle \partial_{u_\alpha}, \partial_{u_\beta} \rangle_U = \delta_{\alpha\beta}$, $\langle \partial_{v_i}, \partial_{u_\alpha} \rangle = 0$, δ_{ij} is a usual Kronecker symbol and $\varepsilon_j = -1$ for $1 \leq j \leq p$ and $\varepsilon_j = 1$ for $p+1 \leq j \leq p+q = n$. Denote by A_{ij}^α the analog of Clifford coefficients, i. e. the coefficients in the decomposition

$$\eta j(\partial_{u_\alpha}) \partial_{v_i} = \sum_{j=1}^n A_{ij}^\alpha \partial_{v_j} \in V.$$

Denote by B_{ij}^β the structure constants in

$$[\partial_{v_i}, \partial_{v_j}] = \sum_{\beta=1}^m B_{ij}^\beta \partial_{u_\beta} \in U.$$

They remain the same as in case of H -type groups with sub-Riemannian metric. The structure constants B_{ij}^β are connected to the analog of Clifford coefficients in a following way:

Lemma 1.

$$B_{ij}^\alpha = \varepsilon_j A_{ij}^\alpha,$$

Proof. We calculate

$$(3.2) \quad \langle \eta j(\partial_{u_\alpha}) \partial_{v_i}, \partial_{v_j} \rangle_V = \langle \partial_{u_\alpha}, [\partial_{v_i}, \partial_{v_j}] \rangle_U = \sum_{\beta=1}^m B_{ij}^\beta \langle \partial_{u_\alpha}, \partial_{u_\beta} \rangle_U = \sum_{\beta=1}^m B_{ij}^\beta \delta_{\alpha\beta} = B_{ij}^\alpha,$$

but from the other side

$$\langle \eta j(\partial_{u_\alpha}) \partial_{v_i}, \partial_{v_j} \rangle_V = \left\langle \sum_{k=1}^n A_{ik}^\alpha \partial_{v_k}, \partial_{v_j} \right\rangle_V = \sum_{k=1}^n A_{ik}^\alpha \langle \partial_{v_k}, \partial_{v_j} \rangle_V = \sum_{k=1}^n A_{ik}^\alpha \varepsilon_k \delta_{kj} = \varepsilon_j A_{ij}^\alpha.$$

□

Let us denote the global coordinates on the group G by $\{v_1, \dots, v_n, u_1, \dots, u_m\}$:

$$\mathbf{v}(\sigma) = \sum_{i=1}^n v_i \partial_{v_i}(e), \quad \mathbf{u}(\sigma) = \sum_{\alpha=1}^m u_\alpha \partial_{u_\alpha}(e).$$

Let us compute the Lie bracket

$$\begin{aligned} [\mathbf{v}(\sigma_1), \mathbf{v}(\sigma_2)] &= \left[\sum_{i=1}^n v_i^1 \partial_{v_i}(e), \sum_{j=1}^n v_j^2 \partial_{v_j}(e) \right] \\ &= v_1^2 \left(\sum_{j=1}^n v_j^1 B_{j1}^1 \partial_{u_1} + \sum_{j=1}^n v_j^1 B_{j1}^2 \partial_{u_2} + \dots + \sum_{j=1}^n v_j^1 B_{j1}^m \partial_{u_m} \right) \\ &\quad + v_2^2 \left(\sum_{j=1}^n v_j^1 B_{j2}^1 \partial_{u_1} + \sum_{j=1}^n v_j^1 B_{j2}^2 \partial_{u_2} + \dots + \sum_{j=1}^n v_j^1 B_{j2}^m \partial_{u_m} \right) \\ &\quad \vdots \\ &\quad + v_n^2 \left(\sum_{j=1}^n v_j^1 B_{jn}^1 \partial_{u_1} + \sum_{j=1}^n v_j^1 B_{jn}^2 \partial_{u_2} + \dots + \sum_{j=1}^n v_j^1 B_{jn}^m \partial_{u_m} \right). \end{aligned}$$

Therefore, the left-invariant vector fields obtained by the rule $V_i(\sigma) = dL_\sigma \partial_{v_i}(e)$ are as follows

$$V_i = \partial_{v_i} + \frac{1}{2} \sum_{\alpha=1}^m \left(\sum_{j=1}^n v_j B_{ji}^\alpha \right) \partial_{u_\alpha}, \quad i = 1, \dots, n$$

and

$$U_\alpha = \partial_{u_\alpha}, \quad \alpha = 1, \dots, m.$$

The vector fields remain the same as in sub-Riemannian case when the metric on V is positively definite, since the matrix dL_σ is obtained from the group structure, which we do not change (compare with [11]). Matrices $B^\alpha = (B_{ji}^\alpha)$, $\alpha = 1, \dots, m$, $i, j = 1, \dots, n$, are in one-to-one correspondance defined by (3.2) with matrices $j_\alpha = j(\partial_{u_\alpha})$ that are the standard generators of representation for Clifford algebra $\text{Cl}(U, m)$. Analogously to sub-Riemannian case we can write the horizontal gradient, i. e. the vector $\mathcal{V} = (V_1, \dots, V_n)$, in the form

$$\mathcal{V} = \nabla_v + \frac{1}{2} \left(B^1 v \partial_{u_1} + \dots + B^m v \partial_{u_m} \right),$$

with $v = (v_1, \dots, v_n)$ and $\nabla_v = (\partial_{v_1}, \dots, \partial_{v_n})$.

Remark. Notice that $\{V_i, U_\alpha\}$, $i = 1, \dots, n$, $\alpha = 1, \dots, m$ is an orthonormal basis with respect to the left-invariant metric $\langle \cdot, \cdot \rangle$.

Proof. By definition of metric $\langle \cdot, \cdot \rangle$ on algebra \mathcal{G} we have:

$$\begin{aligned} \langle V_i, V_j \rangle &= \langle V_i, V_j \rangle_V = \langle \partial_{v_i}, \partial_{v_j} \rangle_V = \varepsilon_j \delta_{ij}, \\ \langle U_\alpha, U_\beta \rangle &= \langle U_\alpha, U_\beta \rangle_U = \langle \partial_{u_\alpha}, \partial_{u_\beta} \rangle_U = \delta_{\alpha\beta}, \\ \langle V_i, U_\alpha \rangle &= 0. \end{aligned}$$

□

The connection on a group G is defined by Koszul formula ([18])

$$\langle \nabla_X Y, Z \rangle = \frac{1}{2} \left(X \langle Y, Z \rangle + Y \langle Z, X \rangle - Z \langle X, Y \rangle - \langle X, [Y, Z] \rangle + \langle Y, [Z, X] \rangle + \langle Z, [X, Y] \rangle \right).$$

For the orthonormal basis of \mathcal{G} this formula receives the form

$$\langle \nabla_X Y, Z \rangle = \frac{1}{2} \left(- \langle X, [Y, Z] \rangle + \langle Y, [Z, X] \rangle + \langle Z, [X, Y] \rangle \right),$$

where X, Y, Z are arbitrary basis elements in the corresponding Lie algebra $\mathcal{G} = U \oplus V$. In the case of H -type group we calculate that

$$\begin{aligned} \langle \nabla_{V_i} V_j, U_\alpha \rangle &= \frac{1}{2} \langle U_\alpha, [V_i, V_j] \rangle_U = \frac{1}{2} \langle \eta j(U_\alpha) V_i, V_j \rangle_V, \\ \langle \nabla_{U_\alpha} V_i, V_j \rangle &= -\frac{1}{2} \langle U_\alpha, [V_i, V_j] \rangle_U = -\frac{1}{2} \langle \eta j(U_\alpha) V_i, V_j \rangle_V, \\ \langle \nabla_{V_i} U_\alpha, V_j \rangle &= \frac{1}{2} \langle U_\alpha, [V_j, V_i] \rangle_U = -\frac{1}{2} \langle U_\alpha, [V_i, V_j] \rangle_U = -\frac{1}{2} \langle \eta j(U_\alpha) V_i, V_j \rangle_V, \end{aligned}$$

for any $V_i, V_j \in V$, $i \neq j$, and $U_\alpha \in U$. It is easy to verify that all the other combinations result as 0. From here we obtain that the connection is given by

$$\begin{aligned} \nabla_{V_i} V_j &= \frac{1}{2} [V_i, V_j], \\ \nabla_{U_\alpha} V_i &= \nabla_{V_i} U_\alpha = -\frac{1}{2} \eta j(U_\alpha) V_i, \\ \nabla_{U_\alpha} U_\beta &= 0. \end{aligned} \tag{3.3}$$

4. GEODESICS

In this section we compute geodesics passing through the identity with a given initial vector. By geodesic we mean a curve $\gamma: [0, 1] \rightarrow G$ which satisfies the geodesic equation $\nabla_{\dot{\gamma}} \dot{\gamma} = 0$. We define functions $t \mapsto v(t)$ and $t \mapsto u(t)$, such that $\gamma(t) = (v(t), u(t))$ and $\dot{v}(t)$ and $\dot{u}(t)$ are the projections of $\dot{\gamma}(t)$ onto V and U respectively.

Then in terms of global coordinates on G one gets $\gamma(t) = (v_1(t), \dots, v_n(t), u_1(t), \dots, u_m(t))$ and the derivative with respect to t is

$$\begin{aligned} \dot{\gamma} &= \sum_{i=1}^n \dot{v}_i V_i + \sum_{\alpha=1}^m \dot{u}_\alpha U_\alpha = \sum_{i=1}^n \dot{v}_i \left(\partial_{v_i} + \frac{1}{2} \sum_{\alpha=1}^m \left(\sum_{j=1}^n \partial_{v_j} B_{ji}^\alpha \right) \partial_{u_\alpha} \right) + \sum_{\alpha=1}^m \dot{u}_\alpha \partial_{u_\alpha} \\ (4.1) \quad &= \sum_{i=1}^n \dot{v}_i \partial_{v_i} + \sum_{\alpha=1}^m \left(\dot{u}_\alpha + \frac{1}{2} \sum_{i,j=1}^n v_j B_{ji}^\alpha \dot{v}_i \right) \partial_{u_\alpha} = \dot{v} + \left(\dot{u} + \frac{1}{2} [\dot{v}, v] \right). \end{aligned}$$

Therefore,

$$(4.2) \quad \nabla_{\dot{\gamma}} \dot{\gamma} = \nabla_{\dot{\gamma}} \dot{v} + \nabla_{\dot{\gamma}} \left(\dot{u} + \frac{1}{2} [\dot{v}, v] \right).$$

Let us compute the first term in right hand side of (4.2)

$$\begin{aligned} \nabla_{\dot{\gamma}} \dot{v} &= \nabla_{\dot{v} + \dot{u} + \frac{1}{2} [\dot{v}, v]} \dot{v} = \nabla_{\dot{v}} \dot{v} + \nabla_{\dot{u}} \dot{v} + \frac{1}{2} \nabla_{[\dot{v}, v]} \dot{v} = \ddot{v} - \frac{1}{2} \eta j(\dot{u}) \dot{v} - \frac{1}{4} \eta j([\dot{v}, v]) \dot{v} \\ &= \ddot{v} - \frac{1}{2} \eta j(\dot{u} + \frac{1}{2} [\dot{v}, v]) \dot{v}. \end{aligned}$$

Analogously, we calculate the second term

$$\nabla_{\dot{\gamma}} \left(\dot{u} + \frac{1}{2} [\dot{v}, v] \right) = \ddot{u} + \frac{1}{2} [\ddot{v}, v] - \frac{1}{2} \eta j(\dot{u} + \frac{1}{2} [\dot{v}, v]) \dot{v}.$$

Therefore,

$$\nabla_{\dot{\gamma}} \dot{\gamma} = \left(\ddot{v} - \eta j(\dot{u} + \frac{1}{2} [\dot{v}, v]) \dot{v} \right) + \left(\ddot{u} + \frac{1}{2} [\ddot{v}, v] \right)$$

and the geodesic γ is characterised by the system of equations:

$$\begin{aligned} (4.3) \quad &\ddot{v} - \eta j(\dot{u} + \frac{1}{2} [\dot{v}, v]) \dot{v} = 0, \\ &\ddot{u} + \frac{1}{2} [\ddot{v}, v] = 0. \end{aligned}$$

From the second equation of this sytem we can deduce that $\dot{u} + \frac{1}{2} [\dot{v}, v] = \text{constant}$. It is enough to describe geodesics which pass through 0, since the left translation of every geodesic is again geodesic. Let us assume the following initial conditions for γ :

$$\gamma(0) = 0, \quad \dot{\gamma}(0) = \dot{v}^0 + \dot{u}^0,$$

which implies that

$$\begin{aligned} v(0) &= 0, & u(0) &= 0, \\ \dot{v}(0) &= \dot{v}^0, & \dot{u}(0) &= \dot{u}^0. \end{aligned}$$

Then

$$(4.4) \quad \dot{u} + \frac{1}{2} [\dot{v}, v] = \dot{u}^0 \quad \text{for all } t \in [0, 1]$$

and the first equation of (4.3) takes the form

$$(4.5) \quad \ddot{v} - \eta j(\dot{v}^0)\dot{v} = 0.$$

Observe that this equation coincides with the one for sub-Riemannian geodesics when η is an identity matrix in V [11].

To find a solution $v(t)$ of (4.5) let us first recall some facts from linear algebra.

4.1. Skew-symmetry with respect to semi-Riemannian metric. Recall that $j(u)$ can be written as a real skew-symmetric $(n \times n)$ matrix: $j = -j^t$ if we fix a basis on V . Therefore, it has purely imaginary eigenvalues $\pm i\mu_k$, $\mu_k \in \mathbb{R}$, $k = 1, \dots, \frac{n}{2}$, since n is an even number. It is possible to bring every skew-symmetric matrix to a block diagonal form by an orthogonal transformation. Thus, j can be written in the form

$$(4.6) \quad j = Q\tilde{j}Q^t,$$

where \tilde{j} is a Murnaghan's canonical form of j :

$$\tilde{j} = \begin{pmatrix} 0 & \mu_1 & 0 & \dots & 0 \\ -\mu_1 & 0 & 0 & \dots & 0 \\ 0 & 0 & -\mu_2 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & \mu_{n/2} \\ & & & -\mu_{n/2} & 0 \end{pmatrix}$$

and Q is an orthogonal transformation. From the condition $j = -j^t$ we obtain also that $\tilde{j} = -\tilde{j}^t$. And since $j(u)$ has the property $i)$ of (2.2), all the eigenvalues of j satisfy the condition $\mu_k^2 = |u|^2$, $k = 1, \dots, \frac{n}{2}$.

Denote by A the matrix $\eta j(u)$ from (4.5). The condition $ii)$ of (2.2) yields that the matrix A is skew-symmetric with respect to the scalar product on V and it can be rewritten as $w^t \eta A v + (Aw)^t \eta v = 0$, $\forall v, w \in V$, from where we obtain that

$$(4.7) \quad \eta A + A^t \eta = 0,$$

that is, $A^t = -\eta A \eta$ (skew-symmetric w. r. t. η).

Every square matrix can be represented as a sum of symmetric and skew-symmetric ones, but in our case these matrices are of very special form, that follows from (2.3):

$$(4.8) \quad j = \begin{pmatrix} \Phi_p & \Psi \\ -\Psi^t & \Phi_q \end{pmatrix} = \begin{pmatrix} \Phi_p & 0 \\ 0 & \Phi_q \end{pmatrix} + \begin{pmatrix} 0 & \Psi \\ -\Psi^t & 0 \end{pmatrix} = A_1 + A_2,$$

where Φ_p and Φ_q are matrices of dimensions $p \times p$ and $q \times q$ respectively such that $\Phi_p = -\Phi_p^t$ and $\Phi_q = -\Phi_q^t$; Ψ is a $(p \times q)$ -matrix and, therefore, matrix A_2 is skew-symmetric. In analogous way using (4.7) we may represent the matrix $A = \eta j$ as follows

$$(4.9) \quad A = \begin{pmatrix} -I_p & 0 \\ 0 & I_q \end{pmatrix} \begin{pmatrix} \Phi_p & \Psi \\ -\Psi^t & \Phi_q \end{pmatrix} = \begin{pmatrix} -\Phi_p & -\Psi \\ -\Psi^t & \Phi_q \end{pmatrix} = \begin{pmatrix} -\Phi_p & 0 \\ 0 & \Phi_q \end{pmatrix} + \begin{pmatrix} 0 & -\Psi \\ -\Psi^t & 0 \end{pmatrix} \\ = \tilde{A}_1 + \tilde{A}_2,$$

The essential distinction between matrices j and $A = \eta j$ is that in the decomposition (4.8) of j both matrices A_1 and A_2 are skew-symmetric, but in decomposition (4.9) of A the matrix \tilde{A}_1 is skew-symmetric, but \tilde{A}_2 is symmetric.

The condition (4.7) is the condition for A to be skew-symmetric matrix with respect to $\langle \cdot, \cdot \rangle_V$ and is analogue of skew-symmetry condition (2.3) for positively definite metric. Using it now together with the condition $i)$ of (2.2) we get

$$j^2(u) = \eta A \eta A = -A^t \eta \eta A = -A^t A = -|u|^2 I_V.$$

From another hand,

$$j^2(u) = \eta A \eta A = -\eta A A^t \eta = -|u|^2 I_V,$$

from where it follows that $AA^t = \eta|u|^2 I_V \eta = |u|^2 I_V$. Therefore,

$$(4.10) \quad AA^t = A^t A = |u|^2 I_V.$$

This allows us to rewrite Kaplan's conditions (2.2) as follows. Consider a mapping $A_u: V \rightarrow V$ (we will omit index u for the sake of simplicity), which is assigned to each $u \in U$ and satisfies the following properties:

$$\begin{aligned} i') \quad & AA^t = A^t A = |u|^2 I_V, \\ ii') \quad & \langle Av, w \rangle_V = -\langle v, Aw \rangle_V. \end{aligned}$$

The calculations above show that the conditions $i)$ and $i')$ are equivalent; the condition $ii')$ is just a rewritten form of $ii)$ in terms of matrix A . Therefore, we get an independent definition of sub-semi-Riemannian H -type group.

Another effect of the property (4.10) is that A appears to have n mutually orthogonal unit eigenvectors and is a diagonalisable matrix (over \mathbb{C} in general). Observe that the matrix A^2 is not symmetric, but it satisfies the condition $(A^2)^t = \eta A^2 \eta$ (symmetric with respect to η).

It is also worth mentioning that even though the matrix A is real, the condition $A^t = A^*$ for adjoint matrix A^* is not true in our situation as in the case of positively definite scalar product. We define the adjoint matrix $A^{*,\eta}$ in our case by $\langle Av, w \rangle_V = \langle v, A^{*,\eta} w \rangle_V$. It has the following view $A^{*,\eta} = \eta A^t \eta$, i. e. $A^{*,\eta} = -A$. Then the condition of A being normal holds: $AA^{*,\eta} = A^{*,\eta} A = -A^2$.

Let us write the characteristic polynomial for the matrix A :

$$(4.11) \quad p_A(\lambda) = \lambda^n + b_{n-1}\lambda^{n-1} + \dots + b_2\lambda^2 + b_1\lambda + b_0 = 0,$$

where the coefficients b_k in front of λ^k , $k = 0, \dots, n-1$, are computed by the known formula:

$$b_k = (-1)^k \sum_{i=1}^n A_k^i,$$

where by symbol A_k^i we denote $(k \times k)$ -dimensional principal minor of the matrix A , that is the determinant of the matrix obtained from the matrix A by erasing its $n-k$ diagonal elements. For a $(n \times n)$ -square matrix A there are n $(k \times k)$ -dimensional principal minors for each $k = 1, \dots, n$. Notice that if k is odd, $n-k$ is also odd.

Lemma 2. *All odd-dimensional principal minors of skew-symmetric w. r. t. η matrix A are zero.*

Proof. We calculate

$$\det(A_k^i)^t = \det A_k^i = \det(-\eta(A_k^i)^t \eta) = \det(-(A_k^i)^t) = \det(-I_k) \det(A_k^i)^t = (-1)^k \det(A_k^i)^t,$$

where I_k is a $(k \times k)$ -identity matrix. From here it follows that $\det A_k^i = 0$ for odd k . \square

Therefore, the characteristic polynomial (4.11) receives the following form:

$$(4.12) \quad p_A(\lambda) = \lambda^n + a_{\frac{n}{2}-1}\lambda^{n-2} + \dots + a_1\lambda^2 + a_0 = 0,$$

where $a_k = b_{2k}$, $k = 1, \dots, \frac{n}{2}-1$, and $a_0 = b_0 = \det A$. Observe that $a_0 \neq 0$ since the dimension n of the matrix A is even. Therefore, 0 is not an eigenvalue of A and all eigenvalues of A come in pairs $\pm\lambda_k$, $k = 1, \dots, \frac{n}{2}$.

Lemma 3. *If λ is an eigenvalue for A with eigenvector v , then $-\lambda$ is an eigenvalue for A^t with eigenvector ηv .*

Proof. Notice that $\langle v, w \rangle_V = \langle \eta v, \eta w \rangle_V$ for any two vectors v and w in V . We calculate

$$\langle Av, w \rangle_V = \langle \lambda v, w \rangle_V = \lambda \langle v, w \rangle_V = \lambda \langle \eta v, \eta w \rangle_V, \quad \forall w \in V.$$

But from another hand,

$$\langle Av, w \rangle_V = \langle -\eta A \eta^t v, w \rangle_V = -\langle A^t \eta v, \eta w \rangle_V, \quad \forall w \in V,$$

yielding the desired result. \square

The operator A has n in general complex eigenvalues and the corresponding complex eigenvectors. The following fact is known, see, for example, [19].

Proposition 1. *Let A be a $(n \times n)$ -matrix with real entries. If the equation $\det(A - \lambda I) = 0$ has a complex solution $\lambda = \alpha + i\beta$, $\beta \neq 0$, then there exist nonzero vectors x and y in V such that*

$$Ax = \alpha x - \beta y, \quad Ay = \alpha y + \beta x.$$

In particular, $\text{span}\{x, y\}$ is a nontrivial invariant subspace for A .

Notice that if $\lambda = \alpha + i\beta$ is a solution of characteristic polynomial (4.12) with multiplicity l , then its conjugate $\bar{\lambda} = \alpha - i\beta$ is also solution of (4.12) with multiplicity l . Due to this fact, lemma 3, proposition 1 and the fact that A is diagonalisable over the complex space V , the following decomposition of A is possible: $A = PDP^{-1}$, where P is orthogonal matrix and D is a block-diagonal matrix with (2×2) -blocks of the following possible forms:

$$(4.13) \quad \begin{pmatrix} \lambda_i & 0 \\ 0 & -\lambda_i \end{pmatrix}, \quad i = 1, \dots, s,$$

$$(4.14) \quad \begin{pmatrix} i\nu_j & 0 \\ 0 & -i\nu_j \end{pmatrix}, \quad j = s+1, \dots, r,$$

and (4×4) -blocks of the form

$$(4.15) \quad \begin{pmatrix} \alpha_k & \beta_k & & 0 \\ -\beta_k & \alpha_k & & \\ & & -\alpha_k & -\beta_k \\ 0 & & \beta_k & -\alpha_k \end{pmatrix}, \quad k = 1, \dots, \frac{n}{4} - \frac{r}{2},$$

where the eigenvalues are rearranged in the following order:

- $\pm\lambda_1, \dots, \pm\lambda_s$, for some $s \in \{0, \dots, \frac{n}{2}\}$, are real eigenvalues of A ,
- $\pm\lambda_{s+1} = \pm i\nu_{s+1}, \dots, \pm\lambda_r = \pm i\nu_r$, for some $r \in \{0, \dots, \frac{n}{2}\}$, $r \geq s$, are purely imaginary eigenvalues of A , and
- $\pm\lambda_{r+1} = \pm(\alpha_1 + i\beta_1)$, $\pm\bar{\lambda}_{r+1}, \dots, \pm\lambda_{\frac{n}{4}-\frac{r}{2}} = \pm(\alpha_{\frac{n}{4}-\frac{r}{2}} + i\beta_{\frac{n}{4}-\frac{r}{2}})$, $\pm\bar{\lambda}_{\frac{n}{4}-\frac{r}{2}}$ are complex eigenvalues of A corresponding to $\frac{n}{4} - \frac{r}{2}$ blocks of the dimension (4×4) .

Notice that the integer r is necessarily even. It is natural that (4×4) -dimensional blocks associated with complex eigenvalues $\pm(\alpha \pm i\beta)$ can appear only if the dimension n is bigger or equal to 4.

Observe that for real eigenvalues λ_i the matrix $\begin{pmatrix} \lambda_i & 0 \\ 0 & -\lambda_i \end{pmatrix}$, $i = 1, \dots, s$, is a diagonalised form for the matrix $\begin{pmatrix} 0 & \lambda_i \\ \lambda_i & 0 \end{pmatrix}$; and for purely imaginary eigenvalues $\lambda_j = i\nu_j$ the matrix $\begin{pmatrix} i\nu_j & 0 \\ 0 & -i\nu_j \end{pmatrix}$ is a diagonalised form of $\begin{pmatrix} 0 & \nu_j \\ -\nu_j & 0 \end{pmatrix}$, $j = s+1, \dots, r$.

Notice that even though A is skew-symmetric w. r. t. η , i. e. the condition (4.7) holds, the matrix D does not have this property in general as long as η is not identity matrix.

Lemma 4. *All eigenvalues of A have absolute value $|u|$.*

Proof. Let us calculate the product $AA^t = PDP^{-1}(P^t)^{-1}D^tP^t = PDD^tP^t$. Observe that the matrix DD^t is a diagonal matrix with entries $|\lambda_k|^2$, $k = 1, \dots, \frac{n}{2}$. Then, using the property (4.10), we see that all eigenvalues of A , real and complex ones, have the same absolute values: $|\lambda_k|^2 = |u|^2$, $k = 1, \dots, \frac{n}{2}$. \square

Taking into account previous comments about eigenvalues we can write that

$$\lambda_1 = \dots = \lambda_s = \nu_{s+1} = \dots = \nu_r = |u|,$$

$$s, r \in \{0, \dots, \frac{n}{2}\} \text{ and } \alpha_k^2 + \beta_k^2 = |u|^2, \quad k = 1, \dots, \frac{n}{4} - \frac{r}{2}.$$

Thus, we showed that the matrix D is built from the the following cells:

$$\begin{pmatrix} 0 & |u| \\ |u| & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & |u| \\ -|u| & 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \alpha & \beta & 0 & 0 \\ -\beta & \alpha & 0 & 0 \\ 0 & 0 & -\alpha & -\beta \\ 0 & 0 & \beta & -\alpha \end{pmatrix},$$

where $\alpha^2 + \beta^2 = |u|^2$.

The appearance of one or another cell of these types in the matrix D depends on the index p of the metric g .

Below we show that the number s of pairs of real eigenvalues of A is equal to 1 in the case of odd index p of the metric η , and $s = 0$ in case of p is even. In other words, if index p is even, then there are no real eigenvalues of A , and if p is odd, there are only two real eigenvalues that are of the form $\pm|u|$.

Note. Notice that $\det A = \det D = (-|u|^2)^s (|u|^2)^{r-s} (|u|^2)^{\frac{n}{2}-r} = (-1)^s |u|^n$. But on the other hand from (4.6) we have $\det A = \det(\eta j) = \det(\eta Q \tilde{j} Q^{-1}) = (-1)^p |u|^n$. Thus, if p is odd, then s is odd too, and if p is even, then s is also even. Notice also that $r - s$ has the same parity as s .

Remark. We restrict our consideration only to the case when the index of the metric p is less or equal to $\frac{n}{2}$, because in an opposite situation we can study the case with the metric $-\eta$, which is analogous.

4.2. Matrices j_1, \dots, j_m . We remind that matrix $j(u)$ does not depend on the metric η and is the same as for sub-Riemannian H -type group, i. e. can be written via endomorphisms j_1, \dots, j_m represented by structure constants (3.2) in the following form:

$$j(u) = u_1 j_1 + \dots + u_m j_m, \quad u = (u_1, \dots, u_m) \in U.$$

Each of j_α is, what is called in linear algebra, a one-column and one-row matrix with 1 at the intersection of i -th row and j -th column, -1 at the intersection of j -th row and i -th column, for $i \neq j$ and each $i, j < n$ and zeros everywhere else (see, for example, matrices j_α , $\alpha = 1, \dots, 7$ from Section 5); and satisfies the following properties

$$(4.16) \quad j_\alpha^t = -j_\alpha, \quad j_\alpha^2 = -I_V, \quad j_\alpha j_\beta = -j_\beta j_\alpha, \quad \alpha \neq \beta.$$

The block-diagonal $(n \times n)$ -matrix consisting of blocks of type $\begin{pmatrix} 0 & \pm 1 \\ \mp 1 & 0 \end{pmatrix}$ is one of them, and let us denote it by j_1 for simplicity. Since $\text{Cl}(U, m)$ by definition is the associative algebra freely generated by m anti-commuting square roots of minus unity, one can associate matrices j_1, \dots, j_m with square roots of $-I_V$. Their linear combination represents the corresponding endomorphism $j(u)$. The matrices j_α have n eigenvalues which come in pairs $\pm i$ and, therefore, $\det j_\alpha = 1$ for each $\alpha = 1, \dots, m$.

We remind that $j(u)$ is a $(n \times n)$ -matrix with the following properties:

- $j^t(u) = -j(u)$;
- $j^2(u) = -|u|^2 I_V$;
- $j^{-1}(u) = -\frac{1}{|u|^2} I_V$;
- the eigenvalues of $j(u)$ are all of the form $\pm i|u|$;
- the determinant of $j(u)$ $\det j(u) = |u|^n$.

From here we can catch out the following properties of matrices ηj_α , $\alpha = 1, \dots, m$.

Lemma 5. *Each matrix ηj_α is skew-symmetric w. r. t. η , i. e. $\eta j_\alpha + j_\alpha^t \eta = 0$.*

Proof. The proof is the same as for an analogous statement (4.7) for the matrix $A = \eta j(u)$ in the previous sub-section. \square

Analogous argument as for matrix A can be applied to show that all eigenvalues of matrices ηj_α , $\alpha = 1, \dots, m$, have absolute value 1 and come in pairs $\pm \lambda_k$, $k = 1, \dots, \frac{n}{2}$.

Lemma 6. *All eigenvalues for matrix ηj_α are of the form ± 1 or $\pm i$, for each $\alpha = 1, \dots, m$.*

Proof. If $a + ib$ is an eigenvalue for ηj_α , then by proposition 1 there exist vectors x and y such that $\eta j_\alpha x = ax - by$ and $\eta j_\alpha y = ay + bx$. Thus, on the space spanned by vectors x and y the linear transformation ηj_α has the matrix

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix},$$

which is possible only if $a = 0$ or $b = 0$, since ηj_α is a 1-row and 1-column matrix. Due to the fact that all eigenvalues of ηj_α have absolute value 1, they should be of the form ± 1 or $\pm i$. \square

Lemma 7. *Matrices ηj_α , $\alpha = 1, \dots, m$, anti-commute with each other.*

Proof. Using lemma 5 and anti-commutativity of matrices j_α we calculate for $\alpha \neq \beta$:

$$\eta j_\alpha \eta j_\beta = -\eta j_\alpha j_\beta^t \eta = \eta j_\alpha j_\beta \eta = -\eta j_\beta j_\alpha \eta = \eta j_\beta j_\alpha^t \eta = -\eta j_\beta \eta j_\alpha.$$

\square

Lemma 8. *If the index p of metric η is odd, then each matrix ηj_α has at least two real eigenvalues ± 1 .*

Proof. The proof follows from the fact that $\det(\eta j_\alpha) = (-1)^p = -1$ for the odd p . From another hand, the determinant of matrix ηj_α is equal to product of all eigenvalues of ηj_α that is $\underbrace{(-1^2) \cdot \dots \cdot (-1^2)}_{s \text{ times}} \underbrace{(-i^2) \cdot \dots \cdot (-i^2)}_{\frac{n}{2}-s \text{ times}}$, where $0 \leq s \leq \frac{n}{2}$ is an integer. This product equals to

-1 if and only if there is an odd number s of pairs of real eigenvalues ± 1 . \square

Note that matrix ηj_1 has only 2 real eigenvalues ± 1 if p is odd. For p – even there exists at least one matrix from $\eta j_1, \dots, \eta j_m$ which does not have real eigenvalues, for example, matrix ηj_1 .

Notice also that if the matrix ηj_α has eigenvalues of the form ± 1 and/or $\pm i$, then the matrix $u_\alpha \eta j_\alpha$ has eigenvalues of the form $\pm u_\alpha$ and/or $\pm i u_\alpha$ respectively, where u_α is a real number.

Lemma 9. *If ± 1 are eigenvalues for all of the matrices $\eta j_1, \dots, \eta j_m$, then $\pm|u|$ are eigenvalues for $\eta j(u)$.*

Proof. We will need the following proposition.

Proposition 2. *Given two anti-commuting $(n \times n)$ -matrices A and B with real eigenvalues $\pm a$ and $\pm b$ respectively, their sum $A + B$ has two of its n eigenvalues of the form $\pm\sqrt{a^2 + b^2}$.*

Proof of the proposition. Let V_a be the a -eigenspace of the matrix A . Then by anti-commutativity we find $BV_a \subseteq V_{-a}$. Notice that matrices A and B^2 commute and, therefore, are simultaneously diagonalizable. So there exists an eigenvector $v \in V_a$ with the property $B^2 v = b^2 v$. Then notice that for the vector $w = \frac{1}{b} Bv$ we have the following relations

$$\begin{aligned} Av &= av, & Bv &= bw, \\ Aw &= -aw, & Bw &= bv. \end{aligned}$$

From here it follows that on the space spanned by vectors v and w the linear transformation $A + B$ has the matrix

$$\begin{pmatrix} a & b \\ b & -a \end{pmatrix},$$

which has eigenvectors with eigenvalues $\pm\sqrt{a^2 + b^2}$. \square

Easy induction shows that the same argument applies to the sum of m matrices: if $\{A_\alpha\}_{\alpha=1}^m$ is a set of anti-commuting matrices and each matrix A_α has two real eigenvalues $\pm a_\alpha$, $\alpha = 1, \dots, m$, then the matrix $A_1 + \dots + A_m$ has two real eigenvalues of the form $\pm\sqrt{a_1^2 + \dots + a_m^2}$.

We obtain the statement of Lemma 9 by taking matrices $\eta j_\alpha = A_\alpha$ and numbers $u_\alpha = a_\alpha$, $\alpha = 1, \dots, m$. \square

Thus, Lemmas 5-9 can be summed up in the following theorem.

Theorem 1. *If the index p of the metric η is odd, then there are two real eigenvalues $\pm|u|$ for the matrix $\eta j(u)$. If p is even, then there are no real eigenvalues for $\eta j(u)$ at all.*

Proof. In order to have 2 real eigenvalues $\pm|u|$ for the matrix $A = \eta j(u)$ all matrices ηj_α should have ± 1 as eigenvalues, which is possible only if p is odd. There are only two real eigenvalues for A , since matrix ηj_1 has only two real eigenvalues. If p is even, matrix ηj_1 does not have real eigenvalues, and, therefore, A does not have real eigenvalues either. \square

Further idea is to construct $\frac{n}{2}$ 2-dimensional invariant subspaces of the operator $A: V \rightarrow V$. Remind that V has a basis consisting of p timelike and q spacelike vectors. There are two possible cases here depending on the index p being odd or even. In case p is odd we have

$$V = W_0 \oplus W_1 \oplus \dots \oplus W_{\frac{p-1}{2}} \oplus \tilde{W}_1 \oplus \dots \oplus \tilde{W}_{\frac{q-1}{2}},$$

where the restriction of the metric g on the subspace W_0 $g|_{W_0}$ is Lorentzian, that is it has index 1, $g|_{W_i}$, $i = 1, \dots, \frac{p-1}{2}$, is a negatively definite metric and $g|_{\tilde{W}_j}$, $j = 1, \dots, \frac{q-1}{2}$, is a positively definite metric. In case of even p we have

$$V = W_1 \oplus \dots \oplus W_{\frac{p}{2}} \oplus \tilde{W}_1 \oplus \dots \oplus \tilde{W}_{\frac{q}{2}},$$

where $g|_{W_i}$, $i = 1, \dots, \frac{p}{2}$, is a negatively definite metric and $g|_{\tilde{W}_j}$, $j = 1, \dots, \frac{q}{2}$, is a positively definite metric. The basis of the subspace W_0 consists of one timelike and one spacelike unit vectors, the basis of subspaces W_i , $i \neq 0$, consists of timelike unit vectors and the basis of subspaces \tilde{W}_j consists of spacelike unit vectors.

4.3. Formulae for geodesics. Let us return to the system of equations (4.3). It is equivalent to the system in new basis $\tilde{v} = Pv \in \mathbb{R}^n$

$$(4.17) \quad \ddot{\tilde{v}} = D\dot{\tilde{v}},$$

where $D = P^{-1}AP$ is a block-type matrix with blocks of the form (4.13)–(4.15) and P is orthogonal matrix.

It is convinient to renumerate coordinates in $\tilde{v} = (\tilde{v}_1, \dots, \tilde{v}_n)$ in the following way:

$\tilde{v} = (\tilde{v}_{s_1}, \tilde{v}_{s_2}, \tilde{v}_{s+1}, \tilde{v}_{s+1_2}, \dots, \tilde{v}_{r_1}, \tilde{v}_{r_2}, \tilde{v}_{r+1}, \tilde{v}_{r+1_2}, \tilde{v}_{r+1_3}, \tilde{v}_{r+1_4}, \dots, \tilde{v}_{\frac{n}{4}+\frac{r}{2_1}}, \tilde{v}_{\frac{n}{4}+\frac{r}{2_2}}, \tilde{v}_{\frac{n}{4}+\frac{r}{2_3}}, \tilde{v}_{\frac{n}{4}+\frac{r}{2_4}})$, where first coordinates $\tilde{v}_{s_1}, \tilde{v}_{s_2}$ correspond to real eigenvalues $\pm|u|$ (if any), then coordinates $\tilde{v}_{j_1}, \tilde{v}_{j_2}$, $j = s+1, \dots, r$, corresponding to purely imaginary eigenvalues $\pm i|u|$ (if any), and then coordinates $\tilde{v}_{r+k_1}, \tilde{v}_{r+k_2}, \tilde{v}_{r+k_3}, \tilde{v}_{r+k_4}$, $k = 1, \dots, \frac{n}{4} - \frac{r}{2}$ corresponding to complex eigenvalues $\pm(\alpha_k \pm i\beta_k)$, $\alpha_k^2 + \beta_k^2 = |u|^2$, $k = 1, \dots, \frac{n}{4} - \frac{r}{2}$ (if $\frac{r}{2} < \frac{n}{4}$). Here lowest index in $\tilde{v}_{i_1}, \tilde{v}_{i_2}, \tilde{v}_{i_3}, \tilde{v}_{i_4}$ denotes the number of the coordinate in \tilde{v} associated with the eigenvalues $\lambda_i, -\lambda_i, \bar{\lambda}_i, -\bar{\lambda}_i$ respectively.

Therefore, the system (4.17) splits into $\frac{n}{2}$ independent systems each in 2 variables

$$(4.18) \quad \begin{cases} \ddot{\tilde{v}}_{s_1} = |u|\dot{\tilde{v}}_{s_2}, \\ \ddot{\tilde{v}}_{s_2} = |u|\dot{\tilde{v}}_{s_1}, \\ \ddot{\tilde{v}}_{j_1} = |u|\dot{\tilde{v}}_{j_2}, \\ \ddot{\tilde{v}}_{j_2} = -|u|\dot{\tilde{v}}_{j_1}, & j = s+1, \dots, r, \\ \ddot{\tilde{v}}_{r+k_1} = \alpha_k \dot{\tilde{v}}_{r+k_1} + \beta_k \dot{\tilde{v}}_{r+k_2}, \\ \ddot{\tilde{v}}_{r+k_2} = -\beta_k \dot{\tilde{v}}_{r+k_1} + \alpha_k \dot{\tilde{v}}_{r+k_2}, \\ \ddot{\tilde{v}}_{r+k_3} = -\alpha_k \dot{\tilde{v}}_{r+k_3} - \beta_k \dot{\tilde{v}}_{r+k_4}, & \alpha_k^2 + \beta_k^2 = |u|^2, \\ \ddot{\tilde{v}}_{r+k_4} = \beta_k \dot{\tilde{v}}_{r+k_3} - \alpha_k \dot{\tilde{v}}_{r+k_4}, & k = 1, \dots, \frac{n}{4} - \frac{r}{2}. \end{cases}$$

There are n equations and n variables in (4.18). We are looking for the solution $\tilde{v} = \tilde{v}(t) \in V$, $t \in [-\infty, +\infty]$, $v(0) = 0$ and with initial velocity $\dot{\tilde{v}}(0) = \dot{\tilde{v}}^0$. From the system (4.18) we get

$$\begin{cases} \dot{\tilde{v}}_{s_1} = -c_{s_1}e^{-|u|t} + c_{s_2}e^{|u|t}, & \dot{\tilde{v}}_{s_1}^0 = -c_{s_1} + c_{s_2}, \\ \dot{\tilde{v}}_{s_2} = c_{s_1}e^{-|u|t} + c_{s_2}e^{|u|t}, & \dot{\tilde{v}}_{s_2}^0 = c_{s_1} + c_{s_2}, \\ \dot{\tilde{v}}_{j_1} = c_j \sin(|u|t) + c_{j_2} \cos(|u|t), & \dot{\tilde{v}}_{j_1}^0 = c_{j_2}, \\ \dot{\tilde{v}}_{j_2} = c_j \cos(|u|t) - c_{j_2} \sin(|u|t), & \dot{\tilde{v}}_{j_2}^0 = c_{j_1}, \\ & j = s+1, \dots, r, \end{cases}$$

$$\begin{cases} \dot{\tilde{v}}_{r+k_1} = e^{\alpha_k t} (-c_{r+k_1} \cos \beta_k t + c_{r+k_2} \sin \beta_k t), & \dot{\tilde{v}}_{r+k_1}^0 = -c_{r+k_1} \\ \dot{\tilde{v}}_{r+k_2} = e^{\alpha_k t} (c_{r+k_1} \sin \beta_k t + c_{r+k_2} \cos \beta_k t), & \dot{\tilde{v}}_{r+k_2}^0 = c_{r+k_2}, \\ \dot{\tilde{v}}_{r+k_3} = e^{-\alpha_k t} (c_{r+k_3} \cos \beta_k t - c_{r+k_4} \sin \beta_k t), & \dot{\tilde{v}}_{r+k_3}^0 = c_{r+k_3} \\ \dot{\tilde{v}}_{r+k_4} = e^{-\alpha_k t} (c_{r+k_3} \sin \beta_k t + c_{r+k_4} \cos \beta_k t), & \dot{\tilde{v}}_{r+k_4}^0 = c_{r+k_4}, \end{cases}$$

$$k = 1, \dots, \frac{n}{4} - \frac{r}{2},$$

It follows that

$$\begin{cases} \dot{\tilde{v}}_{s_1}(t) = \dot{\tilde{v}}_{s_1}^0 \cosh(|u|t) + \dot{\tilde{v}}_{s_2}^0 \sinh(|u|t), \\ \dot{\tilde{v}}_{s_2}(t) = \dot{\tilde{v}}_{s_2}^0 \cosh(|u|t) + \dot{\tilde{v}}_{s_1}^0 \sinh(|u|t), \\ \dot{\tilde{v}}_{j_1}(t) = \dot{\tilde{v}}_{j_2}^0 \sin(|u|t) + \dot{\tilde{v}}_{j_1}^0 \cos(|u|t), \\ \dot{\tilde{v}}_{j_2}(t) = \dot{\tilde{v}}_{j_2}^0 \cos(|u|t) - \dot{\tilde{v}}_{j_1}^0 \sin(|u|t), \end{cases}$$

$$j = s+1, \dots, r,$$

$$\begin{cases} \dot{\tilde{v}}_{r+k_1}(t) = \dot{\tilde{v}}_{r+k_2}^0 \sinh \alpha_k t \sin \beta_k t + \dot{\tilde{v}}_{r+k_1}^0 \cosh \alpha_k t \cos \beta_k t \\ \quad + \dot{\tilde{v}}_{r+k_1}^0 \sinh \alpha_k t \cos \beta_k t + \dot{\tilde{v}}_{r+k_2}^0 \cosh \alpha_k t \sin \beta_k t, \\ \dot{\tilde{v}}_{r+k_2}(t) = -\dot{\tilde{v}}_{r+k_1}^0 \sinh \alpha_k t \sin \beta_k t + \dot{\tilde{v}}_{r+k_2}^0 \cosh \alpha_k t \cos \beta_k t \\ \quad + \dot{\tilde{v}}_{r+k_2}^0 \sinh \alpha_k t \cos \beta_k t - \dot{\tilde{v}}_{r+k_1}^0 \cosh \alpha_k t \sin \beta_k t, \\ \dot{\tilde{v}}_{r+k_3}(t) = \dot{\tilde{v}}_{r+k_4}^0 \sinh \alpha_k t \sin \beta_k t + \dot{\tilde{v}}_{r+k_3}^0 \cosh \alpha_k t \cos \beta_k t \\ \quad - \dot{\tilde{v}}_{r+k_3}^0 \sinh \alpha_k t \cos \beta_k t - \dot{\tilde{v}}_{r+k_4}^0 \cosh \alpha_k t \sin \beta_k t, \\ \dot{\tilde{v}}_{r+k_4}(t) = -\dot{\tilde{v}}_{r+k_3}^0 \sinh \alpha_k t \sin \beta_k t + \dot{\tilde{v}}_{r+k_4}^0 \cosh \alpha_k t \cos \beta_k t \\ \quad - \dot{\tilde{v}}_{r+k_4}^0 \sinh \alpha_k t \cos \beta_k t + \dot{\tilde{v}}_{r+k_3}^0 \cosh \alpha_k t \sin \beta_k t, \end{cases}$$

$$k = 1, \dots, \frac{n}{4} - \frac{r}{2}.$$

Finally, we obtain the solution $\tilde{v}(t)$ to the system (4.18)

$$(4.19) \quad \begin{cases} \tilde{v}_{s_1}(t) = \frac{1}{|u|} (\dot{\tilde{v}}_{s_1}^0 \sinh(|u|t) + \dot{\tilde{v}}_{s_2}^0 \cosh(|u|t) - \dot{\tilde{v}}_{s_2}^0), \\ \tilde{v}_{s_2}(t) = \frac{1}{|u|} (\dot{\tilde{v}}_{s_2}^0 \sinh(|u|t) + \dot{\tilde{v}}_{s_1}^0 \cosh(|u|t) - \dot{\tilde{v}}_{s_1}^0), \end{cases}$$

$$(4.20) \quad \begin{cases} \tilde{v}_{j_1}(t) = \frac{1}{|u|} (\dot{\tilde{v}}_{j_1}^0 \sin(|u|t) - \dot{\tilde{v}}_{j_2}^0 \cos(|u|t) + \dot{\tilde{v}}_{j_2}^0), \\ \tilde{v}_{j_2}(t) = \frac{1}{|u|} (\dot{\tilde{v}}_{j_2}^0 \sin(|u|t) + \dot{\tilde{v}}_{j_1}^0 \cos(|u|t) - \dot{\tilde{v}}_{j_1}^0), \end{cases} \quad j = s+1, \dots, r,$$

$$(4.21) \quad \begin{cases} \tilde{v}_{r+k_1}(t) = A_k \sinh \alpha_k t \sin \beta_k t + B_k \cosh \alpha_k t \cos \beta_k t \\ \quad + B_k \sinh \alpha_k t \cos \beta_k t + A_k \cosh \alpha_k t \sin \beta_k t - B_k, \\ \tilde{v}_{r+k_2}(t) = -B_k \sinh \alpha_k t \sin \beta_k t + A_k \cosh \alpha_k t \cos \beta_k t \\ \quad + A_k \sinh \alpha_k t \cos \beta_k t - B_k \cosh \alpha_k t \sin \beta_k t - A_k, \end{cases}$$

$$A_k = \frac{1}{|u|^2} (\alpha_k \dot{\tilde{v}}_{r+k_2}^0 + \beta_k \dot{\tilde{v}}_{r+k_1}^0), \quad B_k = \frac{1}{|u|^2} (\alpha_k \dot{\tilde{v}}_{r+k_1}^0 - \beta_k \dot{\tilde{v}}_{r+k_2}^0),$$

$$\begin{aligned}
(4.22) \quad & \begin{cases} \tilde{v}_{r+k_3}(t) = -C_k \sinh \alpha_k t \sin \beta_k t - D_k \cosh \alpha_k t \cos \beta_k t \\ \quad + D_k \sinh \alpha_k t \cos \beta_k t + C_k \cosh \alpha_k t \sin \beta_k t + D_k, \\ \tilde{v}_{r+k_4}(t) = D_k \sinh \alpha_k t \sin \beta_k t - C_k \cosh \alpha_k t \cos \beta_k t \\ \quad + C_k \sinh \alpha_k t \cos \beta_k t - D_k \cosh \alpha_k t \sin \beta_k t + C_k, \end{cases} \\
& C_k = \frac{1}{|u|^2}(\alpha_k \dot{v}_{r+k_4}^0 + \beta_k \dot{v}_{r+k_3}^0), \quad D_k = \frac{1}{|u|^2}(\alpha_k \dot{v}_{r+k_3}^0 - \beta_k \dot{v}_{r+k_4}^0), \\
& k = 1, \dots, \frac{n}{4} - \frac{r}{2}.
\end{aligned}$$

A curve $\tilde{v}(t)$ has the same causal type as the initial velocity vector \dot{v}^0 ([1]).

Lemma 10. *The projection of the curve $\tilde{v}(t)$ onto the $(\tilde{v}_{s_1}, \tilde{v}_{s_2})$ -plane, is a brunch of the hyperbola with the canonical equation*

$$(4.23) \quad \left(\tilde{v}_{s_1} + \frac{\dot{v}_{s_2}^0}{|u|} \right)^2 - \left(\tilde{v}_{s_2} + \frac{\dot{v}_{s_1}^0}{|u|} \right)^2 = \frac{-(\dot{v}_{s_1}^0)^2 + (\dot{v}_{s_2}^0)^2}{|u|^2}.$$

Proof. From (4.19) we calculate that $\tilde{v}_{s_1}^2(t) - \tilde{v}_{s_2}^2(t) = \frac{2((\dot{v}_{s_1}^0)^2 - (\dot{v}_{s_2}^0)^2)}{|u|^2}(\cosh |u|t - 1)$. This expression can be rewritten as stated in (4.23). \square

Lemma 11. *The projection of the curve $\tilde{v}(t)$ onto the $(\tilde{v}_{j_1}, \tilde{v}_{j_2})$ -plane, $j = s+1, \dots, r$, is a circle with the center $\left(-\frac{\dot{v}_{j_2}^0}{|u|}, \frac{\dot{v}_{j_1}^0}{|u|} \right)$ of radius $\frac{\sqrt{(\dot{v}_{j_1}^0)^2 + (\dot{v}_{j_2}^0)^2}}{|u|}$.*

Proof. From (4.20) we get $\tilde{v}_{j_1}^2(t) + \tilde{v}_{j_2}^2(t) = \frac{4((\dot{v}_{j_1}^0)^2 + (\dot{v}_{j_2}^0)^2)}{|u|^2} \sinh^2 \frac{|u|t}{2}$. This expression leads to

$$\left(\tilde{v}_{j_1} + \frac{\dot{v}_{j_2}^0}{|u|} \right)^2 + \left(\tilde{v}_{j_2} - \frac{\dot{v}_{j_1}^0}{|u|} \right)^2 = \frac{(\dot{v}_{j_1}^0)^2 + (\dot{v}_{j_2}^0)^2}{|u|^2}.$$

\square

Lemma 12. *The projection of the curve $\tilde{v}(t)$ onto the $(\tilde{v}_{r+k_1}, \tilde{v}_{r+k_2})$ -plane, $k = 1, \dots, \frac{n}{4} - \frac{r}{2}$, is a logarithmic spiral with the equation*

$$(4.24) \quad \tilde{v}_{r+k_1}^2(t) + \tilde{v}_{r+k_2}^2(t) = \frac{1}{|u|^2} \left((\dot{v}_{r+k_1}^0)^2 + (\dot{v}_{r+k_2}^0)^2 \right) e^{2\alpha_k t}.$$

Proof. From (4.21) we obtain

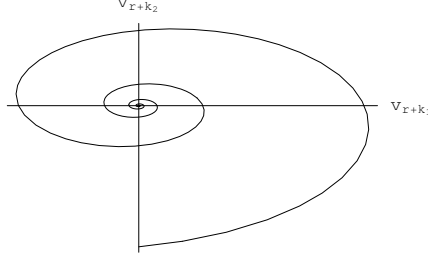
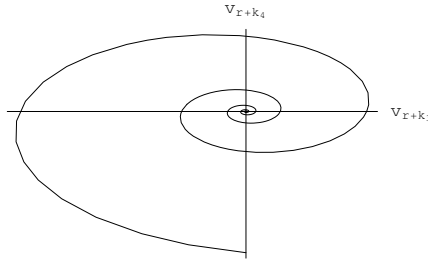
$$\tilde{v}_{r+k_1}^2(t) + \tilde{v}_{r+k_2}^2(t) = \frac{1}{|u|^2} (A_k^2 + B_k^2) e^{2\alpha_k t}.$$

Substituting A_k and B_k into $A_k^2 + B_k^2$ we get the desired formula (4.24). \square

Analogously using (4.22) can be shown the next lemma.

Lemma 13. *The projection of the curve $\tilde{v}(t)$ onto the $(\tilde{v}_{r+k_3}, \tilde{v}_{r+k_4})$ -plane, $k = 1, \dots, \frac{n}{4} - \frac{r}{2}$, is a logarithmic spiral with the equation*

$$(4.25) \quad \tilde{v}_{r+k_3}^2(t) + \tilde{v}_{r+k_4}^2(t) = \frac{1}{|u|^2} \left((\dot{v}_{r+k_3}^0)^2 + (\dot{v}_{r+k_4}^0)^2 \right) e^{-2\alpha_k t}.$$

FIGURE 1. Projection of geodesic onto $(\tilde{v}_{r+k_1}, \tilde{v}_{r+k_2})$ -planeFIGURE 2. Projection of geodesic onto $(\tilde{v}_{r+k_3}, \tilde{v}_{r+k_4})$ -plane

4.4. Solution in a matrix form. Let us rewrite the matrix D (4.13)-(4.15) in the following form $\tilde{D} = \tilde{P}D\tilde{P}^t$, where the matrix \tilde{P} is obtained from the matrix P by rearranging lines in the blocks of type (4.15) and

$$(4.26) \quad \tilde{D} = \begin{pmatrix} |u|D_1 & 0 & 0 & 0 \\ 0 & |u|D_2 & 0 & 0 \\ 0 & 0 & |u|^2D_3 & 0 \\ 0 & 0 & 0 & |u|^2D_4 \end{pmatrix},$$

where D_1 is a $(2s \times 2s)$ -matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, D_2 is a block-diagonal $(2(r-s) \times 2(r-s))$ -matrix consisting of $(r-s)$ (2×2) -blocks of type $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, D_3 is a block-diagonal $((\frac{n}{2}-r) \times (\frac{n}{2}-r))$ -matrix consisting of $\frac{n}{4} - \frac{r}{2}$ (2×2) -blocks of type $D_{3,k} = \begin{pmatrix} \cos \theta_k & \sin \theta_k \\ -\sin \theta_k & \cos \theta_k \end{pmatrix}$, where $\cos \theta_k = \frac{\alpha_k}{|u|^2}$, $k = 1, \dots, \frac{n}{4} - \frac{r}{2}$, and $D_4 = -D_3$. Notice that there is no matrix D_1 in (4.26) if $s = 0$, i. e. if p is even. Analogously, there are no matrices D_1 and D_2 in (4.26) if $r = 0$, and there are no matrices D_3 and D_4 if $k = 0$.

Then

$$(4.27) \quad e^{\tilde{D}t} = \begin{pmatrix} e^{|u|D_1t} & 0 & 0 & 0 \\ 0 & e^{|u|D_2t} & 0 & 0 \\ 0 & 0 & e^{|u|^2D_3t} & 0 \\ 0 & 0 & 0 & e^{|u|^2D_4t} \end{pmatrix}.$$

Notice that

$$D_1^l = \begin{cases} D_1, & l - \text{odd}, \\ I_{2s}, & l - \text{even}, \end{cases} \quad D_2^l = \begin{cases} D_2, & l = 4d + 1, \\ -I_{2(r-s)}, & l = 4d + 2, \\ -D_2, & l = 4d + 3, \\ I_{2(r-s)}, & l = 4d, \quad d = 0, 1, 2, \dots, \end{cases}$$

$$D_{3,k}^l = \begin{pmatrix} \cos l\theta_k & \sin l\theta_k \\ -\sin l\theta_k & \cos l\theta_k \end{pmatrix},$$

$$D_{4,k}^l = -D_{3,k}^l, \quad k = 1, \dots, \frac{n}{4} - \frac{r}{2},$$

where I_{2s} and $I_{2(r-s)}$ are identity matrices of dimensions $(2s \times 2s)$ and $2(r-s) \times 2(r-s)$ respectively.

Taking into account the formulae above we calculate each of the blocks in (4.27)

$$\begin{aligned} e^{|u|D_1 t} &= I_{2s} + D_1|u|t + \frac{D_1^2|u|^2 t^2}{2!} + \frac{D_1^3|u|^3 t^3}{3!} + \dots \\ &= \left(I_{2s} + \frac{(|u|t)^2 I_{2s}}{2!} + \frac{(|u|t)^4 I_{2s}}{4!} + \dots \right) + \left(|u|t D_1 + \frac{(|u|t)^3 D_1}{3!} + \frac{(|u|t)^5 D_1}{5!} + \dots \right) \\ &= \cosh(|u|t) I_{2s} + \sinh(|u|t) D_1, \\ e^{|u|D_2 t} &= I_{2(r-s)} + D_2|u|t + \frac{D_2^2|u|^2 t^2}{2!} + \frac{D_2^3|u|^3 t^3}{3!} + \dots \\ &= \left(I_{2(r-s)} - \frac{(|u|t)^2 I_{2(r-s)}}{2!} + \frac{(|u|t)^4 I_{2(r-s)}}{4!} - \dots \right) + \left(|u|t D_2 - \frac{(|u|t)^3 D_2}{3!} + \frac{(|u|t)^5 D_2}{5!} - \dots \right) \\ &= \cos(|u|t) I_{2(r-s)} + \sin(|u|t) D_2, \\ e^{|u|^2 D_{3,k} t} &= \begin{pmatrix} \sum_{l=0}^{\infty} \frac{\cos l\theta_k}{l!} (|u|^2 t)^l & \sum_{l=0}^{\infty} \frac{\sin l\theta_k}{l!} (|u|^2 t)^l \\ -\sum_{l=0}^{\infty} \frac{\sin l\theta_k}{l!} (|u|^2 t)^l & \sum_{l=0}^{\infty} \frac{\cos l\theta_k}{l!} (|u|^2 t)^l \end{pmatrix} \\ &= \begin{pmatrix} e^{|u|^2 t \cos \theta_k} \cos(|u|^2 t \sin \theta_k) & e^{|u|^2 t \cos \theta_k} \sin(|u|^2 t \sin \theta_k) \\ -e^{|u|^2 t \cos \theta_k} \sin(|u|^2 t \sin \theta_k) & e^{|u|^2 t \cos \theta_k} \cos(|u|^2 t \sin \theta_k) \end{pmatrix} \\ &= e^{\alpha_k t} \begin{pmatrix} \cos \beta_k t & \sin \beta_k t \\ -\sin \beta_k t & \cos \beta_k t \end{pmatrix} \\ e^{|u|^2 D_{4,k} t} &= e^{-\alpha_k t} \begin{pmatrix} \cos \beta_k t & \sin \beta_k t \\ -\sin \beta_k t & \cos \beta_k t \end{pmatrix}, \quad k = 1, \dots, \frac{n}{4} - \frac{r}{2}. \end{aligned}$$

Therefore, $e^{|u|^2 D_3 t}$ is a block-diagonal matrix consisting of $\frac{n}{4} - \frac{r}{2}$ (2×2) -blocks of type $e^{|u|^2 D_{3,k} t}$ and $e^{|u|D_4 t}$ is a block-diagonal matrix consisting of $\frac{n}{4} - \frac{r}{2}$ (2×2) -blocks of type $e^{|u|^2 D_{4,k} t}$.

The solution of the system

$$(4.28) \quad \ddot{\tilde{v}} = \tilde{D} \dot{\tilde{v}},$$

where $\tilde{v} = \tilde{P} \tilde{v}$, with the initial conditions $\tilde{v}(0) = 0$, $\dot{\tilde{v}}(0) = \dot{\tilde{v}}_0$, is of the form $\tilde{v}(t) = (e^{\tilde{D}t} - I_V) \tilde{D}^{-1} \dot{\tilde{v}}_0$. Notice that $\tilde{D}^{-1} = \frac{1}{|u|^2} \tilde{D}^t$, $\tilde{D} \tilde{D}^t = \tilde{D}^t \tilde{D} = |u|^2 I_V$ and $D_1^t = D_1, D_2^t = -D_2$.

Therefore, the solution of (4.28) reduces to

$$(4.29) \quad \tilde{v}(t) = \frac{1}{|u|^2} (e^{\tilde{D}t} - I_V) \tilde{D}^t \dot{\tilde{v}}_0.$$

In order to write the solution more explicitly, let us make the following notations: let $\tilde{v}^1(t)$ denote the $2s$ -dimensional vector consisting of $2s$ first coordinates from the vector $\tilde{v}(t)$, analogously let $\tilde{v}^2(t)$ denote the vector consisting of next $2(r-s)$ coordinates from the vector $\tilde{v}(t)$, let us denote also by $\tilde{v}^{3,k}(t)$ the 2-dimensional part of vector $\tilde{v}(t)$ corresponding to each block $D_{3,k}$, and analogously, $\tilde{v}^{4,k}(t)$ – part of $\tilde{v}(t)$ corresponding to block $D_{4,k}$. In analogous way, let $\dot{\tilde{v}}_0^1$ be the $2s$ -dimensional vector of first $2s$ coordinates of the vector $\dot{\tilde{v}}_0$, $\dot{\tilde{v}}_0^2$ be the vector of next $2(r-s)$ coordinates of the vector $\dot{\tilde{v}}_0$ and let us denote by $\dot{\tilde{v}}_0^{3,k}$ the part of vector $\dot{\tilde{v}}_0$ corresponding to each block $D_{3,k}$, and analogously, $\dot{\tilde{v}}_0^{4,k}$ – part of $\dot{\tilde{v}}_0$ corresponding to block $D_{4,k}$, $k = 1, \dots, \frac{n}{4} - \frac{r}{2}$. Then the solution (4.29) $\tilde{v}(t) = (\tilde{v}^1, \tilde{v}^2, \tilde{v}^{3,1}, \tilde{v}^{4,1}, \dots, \tilde{v}^{3, \frac{n}{4} - \frac{r}{2}}, \tilde{v}^{4, \frac{n}{4} - \frac{r}{2}})(t)$ to the system (4.28) can be written in the form

$$(4.30) \quad \begin{aligned} \tilde{v}^1(t) &= \frac{\dot{\tilde{v}}_0^1}{|u|^2} (\sinh(|u|t) I_{2s} + (\cosh(|u|t) - 1) D_1), \\ \tilde{v}^2(t) &= \frac{\dot{\tilde{v}}_0^2}{|u|^2} (\sin(|u|t) I_{2(r-s)} + (1 - \cos(|u|t)) D_2), \\ \tilde{v}^{3,k}(t) &= \dot{\tilde{v}}_0^{3,k} e^{2\alpha_k t} \mathbb{B}_k, \\ \tilde{v}^{4,k}(t) &= \dot{\tilde{v}}_0^{4,k} e^{-2\alpha_k t} \mathbb{B}_k, \end{aligned}$$

where matrix $\mathbb{B}_k = \frac{1}{|u|^2} \begin{pmatrix} 1 - \cos(\beta_k t) & \sin(\beta_k t) \\ -\sin(\beta_k t) & 1 - \cos(\beta_k t) \end{pmatrix}$, $k = 1, \dots, \frac{n}{4} - \frac{r}{2}$.

For the horizontal part of the geodesic $\gamma(t)$ we make following observations: 2-dimensional $\tilde{v}^1(t)$, corresponding to the cell in \tilde{D} of type D_1 , describes the geodesic which lives in a sub-Lorentzian 3-dimensional Heisenberg group, i. e. Heisenberg group with the metric tensor $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ on the horizontal bundle; 2-dimensional projections of $\tilde{v}^2(t)$, corresponding to the cells in \tilde{D} of type D_2 , describe the geodesics which live in a sub-Riemannian 3-dimensional Heisenberg group with a Riemannian metric $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ on the horizontal bundle. There are no analogues to $\tilde{v}^{3,k}(t)$ or $\tilde{v}^{4,k}(t)$ in Heisenberg group, but together $(\tilde{v}^{3,k}(t), \tilde{v}^{4,k}(t))$ for each k they correspond to the geodesic which lives in a sub-semi-Riemannian H -type Quaternion group with the metric of index 2 on the horizontal subbundle and corresponds to the (4×4) -dimensional block of type

$$\begin{pmatrix} \cos \theta_k & \sin \theta_k & 0 & 0 \\ -\sin \theta_k & \cos \theta_k & 0 & 0 \\ 0 & 0 & -\cos \theta_k & -\sin \theta_k \\ 0 & 0 & \sin \theta_k & -\cos \theta_k \end{pmatrix},$$

(see [13]).

Once $\tilde{v}(t)$ is calculated one can find the horizontal part $v(t)$ of the geodesic $\gamma(t) = (v(t), u(t))$ by

$$(4.31) \quad v = P^{-1} \tilde{P}^{-1} \tilde{v}.$$

To find a vertical part $u(t)$ we substitute $v = P^{-1}\tilde{P}^{-1}\tilde{v}$ into the condition (4.4) and obtain that

$$\dot{u} + \frac{1}{2}[P^{-1}\tilde{P}^{-1}\dot{\tilde{v}}, \tilde{v}] = \dot{u}_0,$$

and, integrating, we get

$$(4.32) \quad u(t) = -\frac{1}{2} \int_0^t [P^{-1}\tilde{P}^{-1}\dot{\tilde{v}}, P^{-1}\tilde{P}^{-1}\tilde{v}] dt + \dot{u}_0 t.$$

5. EXAMPLE

In this section we provide an illustrating example of how the theory above can be applied to concrete situation.

Let us consider the H -type group with algebra $V \oplus U$, $V = \mathbb{R}^8$, $U = \mathbb{R}^7$, and with nondegenerate metric η of arbitrary index $p \leq 4$ on V .

Let $\gamma(t) = (v(t), u(t))$ be a geodesic on such group. It is sufficient for us to find only the part $\tilde{v}(t)$ since $v(t)$ and $u(t)$ can be obtained from it by (4.31) and (4.32).

To escape extra indexes it would be convenient to denote initial vertical velocity \dot{u}^0 by $u = (u_1, \dots, u_7)$. Since $j(u)$ appears from the structure of the algebra and does not depend on the metric on V , we will base on the fact that matrix $j(u)$ for octonion H -type group in sub-Riemannian case is already calculated in [4].

$$j(u) = \begin{pmatrix} 0 & u_1 & u_2 & u_3 & u_4 & u_5 & u_6 & u_7 \\ -u_1 & 0 & u_3 & -u_2 & u_5 & -u_4 & -u_7 & u_6 \\ -u_2 & -u_3 & 0 & u_1 & u_6 & u_7 & -u_4 & -u_5 \\ -u_3 & u_2 & -u_1 & 0 & u_7 & -u_6 & u_5 & -u_4 \\ -u_4 & -u_5 & -u_6 & -u_7 & 0 & u_1 & u_2 & u_3 \\ -u_5 & u_4 & -u_7 & u_6 & -u_1 & 0 & -u_3 & u_2 \\ -u_6 & u_7 & u_4 & -u_5 & -u_2 & u_3 & 0 & -u_1 \\ -u_7 & -u_6 & u_5 & u_4 & -u_3 & -u_2 & u_1 & 0 \end{pmatrix},$$

where $u = (u_1, \dots, u_7) \in \mathbb{R}^7$. Matrices j_α are taken from [4] and have the following forms:

$$j_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad j_2 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \end{pmatrix},$$

$$j_3 = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix}, \quad j_4 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$j_5 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad j_6 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$j_7 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Let j_1, \dots, j_7 be endomorphisms from subsection 4.2 satisfying (4.16) and such that

$$j(u) = u_1 j_1 + \dots + u_7 j_7.$$

Denote as before the matrix $\eta j(u)$ by symbol A . Denote also $|u|^2 = u_1^2 + u_2^2 + u_3^2 + u_4^2 + u_5^2 + u_6^2 + u_7^2$.

Let us find the characteristic polynomials and eigenvalues of A for all $p = 1, \dots, 4$.

If $p = 1$, then

$$p_A^1(\lambda) = \lambda^8 + 2|u|^2 \lambda^6 - 2|u|^6 \lambda^2 - |u|^8,$$

and eigenvalues of A are $\pm|u|, \pm i|u|, \pm i|u|, \pm i|u|$. Notice that all matrices $\eta j_1, \dots, \eta j_7$ have the same eigenvalues $\pm 1, \pm i, \pm i, \pm i$.

We see that if $p = 1$, then $s = 1$, $r = 4$ and $k = 0$ in (4.13)–(4.15) and, therefore, $\tilde{v}(t) = (\tilde{v}^1(t), \tilde{v}^2(t))$, $\tilde{v}(0) = (\tilde{v}_0^1, \tilde{v}_0^2)$. By (4.30) we get

$$\begin{aligned} \tilde{v}^1(t) &= \frac{\tilde{v}_0^1}{|u|^2} (\sinh |u|t I_2 + (\cosh |u|t - 1) D_1), \\ \tilde{v}^2(t) &= \frac{\tilde{v}_0^2}{|u|^2} (\sin |u|t I_6 + (1 - \cos |u|t) D_2), \end{aligned}$$

where D_1, D_2 are (6×6) -matrix described in (4.26), I_2, I_6 – identity matrices.

If $p = 2$, then

$$p_A^2(\lambda) = \lambda^8 + 4u_1^2 \lambda^6 + 2|u|^2 (a^2 + 2(u_1^2 - u_2^2 - \dots - u_7^2)) \lambda^4 + 4|u|^4 u_1^2 \lambda^2 + |u|^8,$$

and eigenvalues are $\pm(iu_1 \pm \sqrt{u_2^2 + \dots + u_7^2}), \pm i|u|, \pm i|u|$. Notice that matrix ηj_1 has eigenvalues $\pm i, \pm i, \pm i, \pm i$ and matrices $\eta j_2, \dots, \eta j_7$ have eigenvalues $\pm 1, \pm 1, \pm i, \pm i$.

We see that if $p = 2$, then $s = 0$, $r = 2$ and $k = 1$, $\alpha_1 = \sqrt{u_2^2 + \dots + u_7^2}$, $\beta_1 = u_1$ in (4.13)–(4.15) and, therefore, $\tilde{v}(t) = (\tilde{v}^2(t), \tilde{v}^{3,1}(t), \tilde{v}^{4,1}(t))$, $\tilde{v}(0) = (\tilde{v}_0^2, \tilde{v}_0^{3,1}, \tilde{v}_0^{4,1})$. By (4.30) we

get

$$\begin{aligned}\tilde{v}^2(t) &= \frac{\dot{\tilde{v}}_0^1}{|u|^2} (\sin |u|t I_4 + (1 - \cos |u|t) D_2), \\ \tilde{v}^{3,1}(t) &= \dot{\tilde{v}}_0^{3,1} e^{2\alpha_1 t} \mathbb{B}_1, \\ \tilde{v}^{4,1}(t) &= \dot{\tilde{v}}_0^{4,1} e^{-2\alpha_1 t} \mathbb{B}_1,\end{aligned}$$

where D_2 are (4×4) -matrix described in (4.26), I_4 is (4×4) -identity matrix and

$$(5.1) \quad \mathbb{B}_1 = \frac{1}{|u|^2} \begin{pmatrix} 1 - \cos \beta_1 t & \sin \beta_1 t \\ -\sin \beta_1 t & 1 - \cos \beta_1 t \end{pmatrix}.$$

If $p = 3$, then

$$p_A^3(\lambda) = \lambda^8 + 2(u_1^2 + u_2^2 + u_3^2 - u_4^2 - \dots - u_7^2)\lambda^6 - 2|u|^4(u_1^2 + u_2^2 + u_3^2 - u_4^2 - \dots - u_7^2)\lambda^2 - |u|^8,$$

and eigenvalues are $\pm|u|, \pm i|u|, \pm(i\sqrt{u_1^2 + u_2^2 + u_3^2} \pm \sqrt{u_4^2 + \dots + u_7^2})$. Notice that $\eta j_1, \eta j_2, \eta j_3$ have eigenvalues $\pm 1, \pm i, \pm i, \pm i$, and matrices $\eta j_4, \dots, \eta j_7$ have eigenvalues $\pm i, \pm 1, \pm 1, \pm 1$.

We see that if $p = 3$, then $s = 1, r = 2$ and $k = 1, \alpha_1 = \sqrt{u_4^2 + \dots + u_7^2}, \beta_1 = \sqrt{u_1^2 + u_2^2 + u_3^2}$, and $\tilde{v}(t) = (\tilde{v}^2(t), \tilde{v}^3(t), \tilde{v}^{3,1}(t), \tilde{v}^{4,1}(t))$, $\tilde{v}(0) = (\dot{\tilde{v}}_0^1, \dot{\tilde{v}}_0^2, \dot{\tilde{v}}_0^{3,1}, \dot{\tilde{v}}_0^{4,1})$. Then, by (4.30)

$$\begin{aligned}\tilde{v}^1(t) &= \frac{\dot{\tilde{v}}_0^1}{|u|^2} (\sinh |u|t I_2 + (\cosh |u|t - 1) D_1), \\ \tilde{v}^2(t) &= \frac{\dot{\tilde{v}}_0^2}{|u|^2} (\sin |u|t I_2 + (1 - \cos |u|t) D_2), \\ \tilde{v}^{3,1}(t) &= \dot{\tilde{v}}_0^{3,1} e^{2\alpha_1 t} \mathbb{B}_1, \\ \tilde{v}^{4,1}(t) &= \dot{\tilde{v}}_0^{4,1} e^{-2\alpha_1 t} \mathbb{B}_1,\end{aligned}$$

where \mathbb{B}_1 has the same form as in (5.1) with $\beta_1 = \sqrt{u_1^2 + u_2^2 + u_3^2}$, D_1 and D_2 are (2×2) -matrices represented in (4.26).

If $p = 4$, then

$$\begin{aligned}p_A^4(\lambda) &= \lambda^8 + 4(u_1^2 + u_2^2 + u_3^2 - u_4^2 - \dots - u_7^2)\lambda^6 \\ &+ 2(|u|^4 + 2(u_1^2 + u_2^2 + u_3^2 - u_4^2 - \dots - u_7^2)^2)\lambda^4 + 4|u|^4(u_1^2 + u_2^2 + u_3^2 - u_4^2 - \dots - u_7^2)\lambda^2 + |u|^8,\end{aligned}$$

and eigenvalues are $\pm(i\sqrt{u_1^2 + u_2^2 + u_3^2} \pm \sqrt{u_4^2 + \dots + u_7^2}), \pm(i\sqrt{u_1^2 + u_2^2 + u_3^2} \pm \sqrt{u_4^2 + \dots + u_7^2})$. Notice that matrices $\eta j_1, \eta j_2, \eta j_3$ have only purely imaginary eigenvalues $\pm i, \pm i, \pm i, \pm i$, and matrices $\eta j_4, \dots, \eta j_7$ have only real eigenvalues $\pm 1, \pm 1, \pm 1, \pm 1$.

We see that is $p = 4$, then $s = 0, r = 0, k = 2$ and $\alpha_1 = \alpha_2 = \sqrt{u_4^2 + \dots + u_7^2}, \beta_1 = \beta_2 = \sqrt{u_1^2 + u_2^2 + u_3^2}$. Then $\tilde{v}(t) = (\tilde{v}^{3,1}(t), \tilde{v}^{4,1}(t), \tilde{v}^{3,2}(t), \tilde{v}^{4,2}(t))$ and initial velocity is $\dot{\tilde{v}}(0) =$

$(\dot{\tilde{v}}_0^{3,1}, \dot{\tilde{v}}_0^{4,1}, \dot{\tilde{v}}_0^{3,2}, \dot{\tilde{v}}_0^{4,2})$. By formulae (4.30) we have

$$\begin{aligned}\tilde{v}^{3,1}(t) &= \dot{\tilde{v}}_0^{3,1} e^{2\alpha_1 t} \mathbb{B}_1, \\ \tilde{v}^{4,1}(t) &= \dot{\tilde{v}}_0^{4,1} e^{-2\alpha_1 t} \mathbb{B}_1, \\ \tilde{v}^{3,2}(t) &= \dot{\tilde{v}}_0^{3,2} e^{2\alpha_1 t} \mathbb{B}_1, \\ \tilde{v}^{4,2}(t) &= \dot{\tilde{v}}_0^{4,2} e^{-2\alpha_1 t} \mathbb{B}_1,\end{aligned}$$

In fact, the author's guess is that for all sub-semi-Riemannian H -type groups satisfying j^2 -condition all complex eigenvalues $\alpha_k + i\beta_k$ are the same $\forall k$ for every fixed index p . Notice that in case p is even there are no real eigenvalues, i. e. $s = 0$ in (4.13), and in case p is odd $\pm|u|$ are the only couple of real eigenvalues, i. e. $s = 1$.

6. POSSIBLE OPEN PROBLEMS

As a continuation of work in this direction it would be interesting to consider the generalization of matrices A to matrices of order $n = \rho^a k$, where ρ is a prime number and k is prime to ρ , and to consider the corresponding analogue of H -type groups. In this case each matrix j_α would represent a p -th root of -1 (see [15]). The number of such matrices in a set of n -rowed matrices that satisfy the conditions

$$j_\alpha^p = -I, \quad j_\alpha j_\beta = -j_\beta j_\alpha, \quad (\alpha \neq \beta),$$

is not more than $2a + 1$.

Another interesting research line could be considering vector spaces U and V both with non-degenerate metrics $\langle \cdot, \cdot \rangle_U$ and $\langle \cdot, \cdot \rangle_V$. The description of corresponding H -type groups would be probably simpler than in the present article, since the construction of representations of Clifford algebras $\text{Cl}(U)$ are less tricky in the case of nondegenerate indefinite product on U .

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